Lecture 2

Example: We want to store $n-$objects in $r-$closets with n_1 objects going in closet 1, n_2 going in closet 2, ... , n_r going in closet r, where

$$\sum_{i=1}^{r} n_i = n.$$

How many ways are there to do this?

Ans. Number of possible ways = $\binom{n}{n1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}...\binom{n-n_1-n_2-n_3-\cdots-n_{r-1}}{n_r}$, which can be simplified as follows

$$\frac{n!}{n_1!n_2!n_3!\cdots n_r!} = \binom{n}{n_1 \cdots n_r}, \tag{1}$$

where Eq.(1) is called Multinomial Coefficients.

1 Conditional Probability

If A and B are events and $\mathbb{P}(B) > 0$. Then the probability of A given B is the following:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}. \tag{2}$$

Example: You roll a fair die. What is $\mathbb{P}(A \mid B)$ if event A represents the appearing of number 6 and event B represents the outcome is even?

Ans.

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1/6}{3/6} = \frac{1}{2}.$$

From Eq.(2), we can conclude that:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\mathbb{P}(B), \tag{3}$$

which will be very useful in studying the concept of law of total probability.

2 Law of total probability

If $B_1 \cdots B_n$ are mutually disjoints such that

$$\bigcup_{i=1}^{n} B_i = \Omega,$$

and

$$\mathbb{P}(B_i) > 0,$$

for $i = 1, 2, \cdots n$. Then the probability of event A is given by

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i). \tag{4}$$

Proof: By utilizing Eq.(3), and starting from the right hand side (R.H.S) of Eq.(4), we have

$$\sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i) = \sum_{i=1}^{n} \mathbb{P}(A \cap B_i).$$

By utilizing the axioms of probability, Demorgan's law and having in mind that $(A \cap B_1)$, $(A \cap B_2) \cdots$ and $(A \cap B_n)$ are disjoints events, we get

$$\begin{aligned}
\sum_{i=1}^{n} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i) &= \sum_{i=1}^{n} \mathbb{P}(A \cap B_i) \\
&= \mathbb{P}\big((A \cap B_1) \cup (A \cap B_2) \cdots \cup (A \cap B_n)\big) \\
&= \mathbb{P}\big(\bigcup_{i=1}^{n} (A \cap B_i)\big) \\
&= \mathbb{P}\big(A \cap \bigcup_{i=1}^{n} (B_i)\big) = \mathbb{P}(A \cap \Omega) = \mathbb{P}(A),
\end{aligned}$$

which is the left hand side (L.H.S) of Eq.(4).

Example: Suppose that a box contains 5 green balls and 8 white balls. A ball is drawn at random and its color is noticed and then it is replaced together with two more balls of the same color. Let G_2 be the event that the second ball is green and let G_3 be the event that the third ball is green. Find the $\mathbb{P}(G_2)$ and $\mathbb{P}(G_3)$.

Ans. Let G_1 be the event that the first drawn ball is green and W_1 be the event that the first drawn ball is white. Then we have

$$\begin{aligned}
\mathbb{P}(G_2) &= \mathbb{P}(G_2 \mid G_1)\mathbb{P}(G_1) + \mathbb{P}(G_2 \mid W_1)\mathbb{P}(W_1) \\
&= \frac{7}{15}.\frac{5}{13} + \frac{5}{15}.\frac{8}{13}.
\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}(G_3) = {} & \mathbb{P}(G_2 \mid G_1G_2)\mathbb{P}(G_1G_2) + \mathbb{P}(G_3 \mid G_1W_2)\mathbb{P}(G_1W_2) + \mathbb{P}(G_3 \mid W_1G_2)\mathbb{P}(W_1G_2) \\
& + \mathbb{P}(G_2 \mid W_1W_2)\mathbb{P}(W_1W_2) \\
= {} & \frac{9}{17}\mathbb{P}(G_2 \mid G_1)\mathbb{P}(G_1) + \frac{7}{17}\mathbb{P}(W_2 \mid G_1)\mathbb{P}(G_1) + \frac{7}{17}\mathbb{P}(G_2 \mid W_1)\mathbb{P}(W_1) \\
& + \frac{5}{17}\mathbb{P}(W_2 \mid W_1)\mathbb{P}(W_1) \\
= {} & \frac{9}{17}\cdot\frac{7}{15}\cdot\frac{5}{13} + \frac{7}{17}\cdot\frac{8}{15}\cdot\frac{5}{13} + \frac{7}{17}\cdot\frac{5}{15}\cdot\frac{8}{13} + \frac{5}{17}\cdot\frac{10}{15}\cdot\frac{8}{13}
\end{aligned}
$$

3 Bayes rule

In a very simple words, we can say that Bayes rule is looking for the probability of the cause of the event, but not the probability of the effect. Under the same conditions as the law of total probability and by utilizing Eq.(**??**), we have

$$\mathbb{P}(B_j \cap A) = \mathbb{P}(B_j \mid A)\mathbb{P}(A) = \mathbb{P}(A \mid B_j)\mathbb{P}(B_j). \tag{5}$$

Therefore from Eq.(5), we have

$$\mathbb{P}(B_j \mid A) = \frac{\mathbb{P}(A \mid B_j)\mathbb{P}(B_j)}{\sum_{i=1}^{n}\mathbb{P}(A \mid B_i)\mathbb{P}(B_i)}.$$

From the previous Example, we can compute $\mathbb{P}(W_1 \mid G_2)$ as follows

$$
\begin{aligned}
\mathbb{P}(W_1 \mid G_2) &= \frac{\mathbb{P}(G_2 \mid W_1)\mathbb{P}(W_1)}{\mathbb{P}(G_2 \mid W_1)\mathbb{P}(W_1) + \mathbb{P}(G_2 \mid G_1)\mathbb{P}(G_1)} \\
&= \frac{\frac{5}{15}\cdot\frac{8}{13}}{\frac{5}{15}\cdot\frac{8}{13} + \frac{7}{15}\cdot\frac{5}{13}}.
\end{aligned}
$$

4 Independence

If A and B are called independent events, then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\cdot\mathbb{P}(B). \tag{6}$$

If $\mathbb{P}(B) > 0$, then Eq.(6) is equivalent to

$$\mathbb{P}(A \mid B) = \mathbb{P}(A).$$

Example: Tossing a coin twice and letting event A denotes that the first toss is head and event B denotes that the second toss is head. Show that these two events are independent?

Ans. Since ı$(A) = 1/2$ and $\mathbb{P}(B) = 1/2$ and $\mathbb{P}(A \cap B) = 1/4$. Therefore we have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Therefore events A and B are independent.

Example: A dart is thrown repeatedly at a target and the probability at the center is 0.05. If the dart us thrown $n-$times and trials are independent. What is the $\mathbb{P}(A)$ if event A represents at least one dart is thrown at the center?

Ans. Let A^c be the event that you do not achieve at least one at the center. Therefore

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^c) = 1 - (0.95)^n,$$

where $\mathbb{P}(A) = 1$ as n goes to infinity.